

ON THE OPERATOR ALGEBRA FOR THE SPACE-TIME UNCERTAINTY RELATIONS

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1. INTRODUCTION

The purpose of this note is to show that the construction of the C^* -algebra for the space-time uncertainty relations which was introduced by Doplicher, Fredenhagen and Roberts [2,3,4] fits comfortably into the deformation quantization framework developed in [5]. This has the mild advantages that one can work directly with functions on space-time rather than with their Fourier transforms, the treatment of the unbounded space-time operators is fairly smooth, and the sense in which one has a deformation of commutative space-time is made technically precise. Possibly our techniques will be useful in some related situations.

We will not repeat here the physical motivation and the treatment of the uncertainty relations themselves given in [2,3,4]. We will deal only with the construction of the C^* -algebra and the affiliated operators having the desired properties. To the extent that there is no additional complication, we will work in slightly greater generality than immediately needed, both because this might be useful at some later time, and because it clarifies which aspects of structure are involved.

We now recall from [4] the desired mathematical properties, in the form most convenient for our purposes. Let M_0 denote Minkowski space, and let L denote the full Lorentz group. Let L act on 4×4 matrices by similarity, that is, $\Lambda \in L$ sends the matrix σ to $\Lambda\sigma\Lambda^t$ (where t = transpose). Let σ_0 denote the standard symplectic matrix (as in 3.28 of [4]), and let Σ denote its orbit under L . Then Σ is a manifold consisting of certain invertible skew-symmetric matrices, and is a homogeneous space for L . (See 3.24 of [4].) We want unbounded self-adjoint operators \mathbf{q}_μ , $\mu = 0, \dots, 3$, corresponding to the standard coordinates on space-time. Let

$$Q_{\mu\nu} = -i[\mathbf{q}_\mu, \mathbf{q}_\nu],$$

or, more precisely, the closure of the commutator. We want the $Q_{\mu\nu}$'s to be self-adjoint operators which commute with each other and with the \mathbf{q}_μ 's. The

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$Q_{\mu\nu}$'s, while unbounded, will together "generate" a commutative C^* -algebra, their joint spectrum, and will correspond to unbounded real-valued continuous functions on this spectrum. Given a point, s , of the spectrum, each $Q_{\mu\nu}$ can be evaluated at s , and the resulting matrix, $Q_{\mu\nu}(s)$, will be a 4×4 matrix which is skew-symmetric because of how the $Q_{\mu\nu}$'s are defined. We require that this matrix be in Σ . In fact we require (as in [4]) that Σ be the joint spectrum of the $Q_{\mu\nu}$'s, so that each $Q_{\mu\nu}$ is identified with the function on Σ whose value at $\sigma \in \Sigma$ is just $\sigma_{\mu\nu}$.

Finally, we require that all of this have an integrated Weyl form in terms of one-parameter unitary groups generated by the \mathbf{q}_μ 's (i.e. be a "regular realization" in the terminology of [4]). In fact, our objective is to construct a C^* -algebra and specific affiliated elements such that its non-degenerate representations give exactly all the regular realizations.

2. THE C^* -ALGEBRA

To simplify notation we will work largely in a coordinate-free manner. Let V be any finite dimensional real vector space (eventually to be Minkowski space M_0). Let V' denote its vector-space dual, so that each non-zero element of V' can be viewed as an unbounded function on V (a coordinate function for some basis of V). Following the usage in [4] we will sometimes denote elements of V by the letter q , and elements of V' by the letters α and β , but we will also find it convenient to use x or v for elements of V , and p for elements of V' .

Let $GL(V)$ denote the group of invertible operators on V , and let G denote a closed subgroup of $GL(V)$ (eventually the Lorentz group for some Lorentz metric on V). Let J be an invertible operator from V' to V which is skew-symmetric in the sense that $J^t = -J$. (Eventually J will correspond to the standard symplectic matrix σ_0 above.) Now $GL(V)$ acts by similarity on operators from V' to V . We let Σ denote the orbit of J under G , that is, the set of all operators of the form TJT^t for $T \in G$. Then Σ is a homogeneous space for G , and is in particular a manifold (perhaps not connected).

Our C^* -algebra will be a deformation of the commutative C^* -algebra $C_\infty(\Sigma \times V)$ of continuous complex-valued functions on $\Sigma \times V$ vanishing at infinity. Specifically, for suitable functions in $C_\infty(\Sigma \times V)$ we want to define their deformed product by

$$(f \times g)(\sigma, q) = \int_V \int_{V'} f(\sigma, q - \sigma(p)) g(\sigma, q - v) e(v \cdot p) dv dp,$$

where e denotes the function $e(t) = \exp(2\pi it)$, where $v \cdot p$ denotes the pairing between V and V' (which sometimes we will find more convenient to write as $p(v)$), and where we have chosen a Haar (i.e. Lebesgue) measure on V and the corresponding Plancherel Lebesgue measure on V' . This product is a general version of the Weyl, or Moyal, product. We remark that by including the 2π in our definition of e , we are, in effect, choosing to follow the convention in [5] rather than that in [4].

But the above definition of deformed product does not quite fit into the framework of [5] because, while the action of V in [5] is permitted to be very complicated, σ is supposed to be held fixed, while in the above integral σ is

allowed to vary while the action is essentially just translation. One can certainly extend the general framework of [5] to cover this slightly different situation, but this note is not the place to do that. Instead we will show here that we can rearrange matters a little so that the framework of [5] does apply to yield the desired construction. To do this we will initially work on $G \times V$ rather than $\Sigma \times V$, and then move to $\Sigma \times V$ near the end.

Actually, we will see that it is technically convenient for us to work more generally on $E \times V$ where E is any fixed open subset of G . This is because when we consider the unbounded operators affiliated with the C^* -algebra we construct, it will be convenient for some steps to take as E a bounded (open) subset of G .

For fixed E , define an action τ of V on $E \times V$ by

$$\tau_x(T, q) = (T, q + Tx).$$

Let τ also denote the corresponding action on the C^* -algebra $C_b(E \times V)$ of bounded continuous functions on $E \times V$. This action is not strongly continuous. Let $B = C_u(E, V)$ denote the C^* -subalgebra of all strongly continuous vectors for τ , that is, functions f such that $x \mapsto \tau_x f$ is continuous on V for the supremum norm on $C_b(E \times V)$. (The subscript “ u ” stands for “uniformly continuous”.) This puts us directly in the framework of [5,6], so that we can construct corresponding deformed C^* -algebras. Let B^∞ denote the dense $*$ -subalgebra of smooth (i.e. infinitely differentiable) vectors in B . Then for $f, g \in B^\infty$ we can define their deformed product (see page 83 of [6]) by

$$f \times_J g = \int_{V'} \int_V \tau_{Jp}(f) \tau_v(g) e(v \cdot p) dv dp.$$

The integrand will usually not be integrable, and so the integral must be interpreted as an oscillatory integral by means of the factor $e(v \cdot p)$, as discussed early in [5]. For any $(T, q) \in E \times V$ the above formula gives, after using propositions 1.13 or 2.12 of [5],

$$(f \times_J g)(T, q) = \int_{V'} \int_V f(T, q - TJT^t p) g(T, q - v) e(v \cdot p) dv dp.$$

The natural involution is just complex conjugation, $f^*(T, q) = (f(T, q))^-$. We will denote by B_J^∞ the vector space B^∞ equipped with the above deformed product and involution. To view this explicitly as a one-parameter deformation of the pointwise product, we should (as discussed in [5]) simply put a Planck’s constant \hbar in front of J , or more appropriately in the present context, a Planck length λ_p .

Following [5], we put an operator norm on B_J^∞ as follows. Let \mathcal{S}^B denote the vector-space of B -valued Schwartz functions on V , with B -valued inner-product defined by

$$\langle \xi, \eta \rangle_B = \int_V (\xi(v))^* \eta(v) dv.$$

Define a (real-valued) norm on \mathcal{S}^B by

$$\|\xi\| = \|\langle \xi, \xi \rangle_B\|^{1/2},$$

where the norm on the right is that of B . For $f \in B^\infty$ define an operator, L_f , on \mathcal{S}^B by

$$(L_f \xi)(x) = \int_{V'} \int_V \tau_{x+Jp}(f) \xi(x+v) e(v \cdot p) dv dp.$$

It is seen in theorem 4.1 of [Rfd] that L_f is a bounded operator on \mathcal{S}^B , that $f \mapsto L_f$ is a faithful $*$ -representation of B_J^∞ on \mathcal{S}^B for the B -valued inner-product, and that the corresponding norm on B_J^∞ is a C^* -algebra norm. We denote by B_J the C^* -algebra obtained by completing B_J^∞ for this norm. It is represented on the completion of \mathcal{S}^B for its norm. Every state on B can be composed with the B -valued inner-product on \mathcal{S}^B to give an ordinary inner-product on \mathcal{S}^B , and hence a representation of B_J on an ordinary Hilbert space. We obtain in this way a faithful family of ordinary representations of B_J .

Let $A = C_\infty(E \times V)$. Then A is an essential ideal in B which is carried into itself by τ . Then we have A^∞ , A_J^∞ , and A_J , much as above. Moreover, A_J is an essential ideal in B_J by proposition 5.9 of [5], so that we can view B_J as a (unital) subalgebra of the multiplier algebra, $M(A_J)$, of A_J . It is A_J which, for V Minkowski space and $E = G$ the Lorentz group, will almost be our C^* -algebra for the space-time uncertainty relations. (We will still have to move the situation to Σ .)

3. THE AFFILIATED SPACE-TIME OPERATORS

Let $\alpha \in V'$. We want to associate with α an unbounded operator, \mathbf{q}_α , affiliated with A_J in the sense of Baa [1] and Woronowicz [8]. The C^* -algebra “generated” by \mathbf{q}_α should be isomorphic to $C_\infty(\mathbb{R})$, and should consist simply of the range of the map from $C_\infty(\mathbb{R})$ to functions on $E \times V$ which sends $\varphi \in C_\infty(\mathbb{R})$ to $\varphi \circ \alpha$ independent of the E -variable. But there is no action of V on \mathbb{R} for which this map is equivariant (for τ). Since we need equivariance in order to invoke the functoriality of the construction of [5], we enlarge the domain as follows. Consider $C_b(E \times \mathbb{R})$, and define an action, ρ^α , of V on $C_b(E \times \mathbb{R})$ by

$$(\rho_x^\alpha \psi)(T, r) = \psi(T, r - \alpha(Tx)).$$

Define a homomorphism, Φ^α , from $C_b(E \times \mathbb{R})$ into $C_b(E \times V)$ by

$$(\Phi^\alpha(\psi))(T, q) = \psi(T, \alpha(q)).$$

Then it is easily verified that Φ^α is equivariant for ρ^α and τ . Much as above, let $C_u^\alpha(E \times \mathbb{R})$ denote the algebra of strongly continuous vectors for ρ^α , so that its deformation quantization, $(C_u^\alpha(E \times \mathbb{R}))_J$, is defined. Then Φ^α carries $C_u^\alpha(E \times \mathbb{R})$ into $C_u(E \times V)$, and so by the functoriality of our deformation quantization construction (theorem 5.7 of [5]) Φ^α determines a $*$ -homomorphism, Φ_J^α , from $(C_u^\alpha(E \times \mathbb{R}))_J$ to B_J .

Let us now calculate the deformed product on $C_u^\alpha(E \times \mathbb{R})_J$, and see that the product is in fact unchanged—no surprise in view of the basically one-dimensional nature of the action ρ^α . If ψ_1, ψ_2 are ρ^α -smooth vectors in $C_u(E \times \mathbb{R})$, then much as for B^∞ earlier

$$(\psi_1 \times_J \psi_2)(T, r) = \int_{V'} \int_V \psi_1(T, r - \alpha(TJT^t p)) \psi_2(T, r - \alpha(v)) e(v \cdot p) dv dp.$$

But this “sees” only the α -component of $v \in V$, and so by proposition 1.3 of [6], which is just a reinterpretation of proposition 1.11 of [5], we can reduce the domain of integration as follows. Let W denote the kernel of α in V , so that W^\perp is just the linear span of α in V' . Then the above integral becomes

$$\int_{W^\perp} \int_{V/W} \psi_1(T, r - \alpha(TJT^t p)) \psi_2(T, r - \alpha(v)) e(v \cdot p) dv dp.$$

But since p is now running over the span of α and TJT^t is skew-symmetric, $\alpha(TJT^t p) = 0$. From corollary 1.12 of [5] (basically the Fourier inversion formula) it follows that

$$\psi_1 \times_J \psi_2 = \psi_1 \psi_2,$$

the pointwise product. The C^* -completion is then evidently $C_u(E \times \mathbb{R})$ itself.

Thus we see that Φ^α determines a homomorphism, Φ_J^α , of the commutative C^* -algebra $C_u(E \times \mathbb{R})$ into B_J . Since Φ^α for the undeformed algebras is injective, it follows from proposition 5.8 of [5] that Φ_J^α is injective.

Notice, however, that Φ^α does not carry $C_\infty(E \times \mathbb{R})$ into $C_\infty(E \times V)$ (unless V is one-dimensional) since functions in its range are constant on the kernel of α . But Φ^α will give a “morphism” from $C_\infty(E \times \mathbb{R})$ to $C_\infty(E \times V)$ in the sense of Woronowicz [8]. This means that Φ^α carries $C_\infty(E \times \mathbb{R})$ into the multiplier algebra, $C_b(E \times V)$, of $C_\infty(E \times V)$, and that $\Phi^\alpha(C_\infty(E \times \mathbb{R}))C_\infty(E \times V)$ (linear span) is dense in $C_\infty(E \times V)$. By the functoriality of the deformation quantization construction with respect to morphisms, given in theorem 3.1 of [7], it follows that when we restrict Φ_J^α to $C_\infty(E \times \mathbb{R})$ it determines a morphism to A_J .

We are now in a position to define the operators \mathbf{q}_α (which will be the space-time operators when V is Minkowski space). Given $\varphi \in C_\infty(\mathbb{R})$, view it as the corresponding function on $E \times \mathbb{R}$ independent of the E -variable. Although the resulting function need not be in $C_\infty(E \times \mathbb{R})$, we clearly obtain in this way a morphism from $C_\infty(\mathbb{R})$ to $C_\infty(E \times \mathbb{R})$. But morphisms can be composed—see the comments near the end of section 0 of [W]. So if we compose the above morphism with Φ_J^α , we obtain a morphism, Θ_J^α , from $C_\infty(\mathbb{R})$ to A_J . The unbounded function $\iota(t) = t$ on \mathbb{R} defines a self-adjoint operator affiliated with $C_\infty(\mathbb{R})$, in the sense of [W]. But according to theorem 1.2 of [W] affiliated operators are transported by morphisms, and so there is a well-defined self-adjoint operator, $\Theta_J^\alpha(\iota)$, affiliated with A_J . We set

$$\mathbf{q}_\alpha = \Theta_J^\alpha(\iota).$$

The one-parameter unitary group, v_t , generated by the operator ι affiliated to $C_\infty(\mathbb{R})$ is clearly $v_t(r) = e(tr)$ (up to conventions about 2π). Under the morphism Θ_J^α this is carried to the one-parameter unitary group, u_t^α , generated by \mathbf{q}_α . Since we can scale α , we only need $u_1^\alpha = e(\mathbf{q}_\alpha)$, which we denote by u_α . We will show that the u_α ’s satisfy the appropriate Weyl relations.

All the above is very natural, but there is an interesting technical point which now needs to be emphasized, and which is responsible for our working in the generality of the subsets E of G . Suppose $\varphi \in C_\infty(\mathbb{R})$ is given, and view it as a

function on $E \times \mathbb{R}$ as done above. Then this function need not be in $C_u^\alpha(E \times \mathbb{R})$, because for $x \in V$ we have

$$\|\rho_x^\alpha(\varphi) - \varphi\| = \sup\{|\varphi(r - \alpha(Tx)) - \varphi(r)| : T \in E \text{ and } r \in \mathbb{R}\},$$

which need not go to 0 as $x \rightarrow 0$, since the set E over which T varies may be an unbounded subset of $GL(V)$. Then even if φ is a smooth vector in $C_\infty(\mathbb{R})$ for translation, $\Theta^\alpha(\varphi)$ need not be a strongly continuous vector in $C_b(E \times V)$, and so its deformed product with elements of A^∞ can not be computed by the oscillatory integral used earlier. A very similar situation occurs in [7]—see the comments after definition 3.4 of [7]. These include comments about how to compute the deformed product by an approximation process, and analogous comments apply here.

However, a simple argument shows that if E is a *bounded* subset of $GL(V)$ then indeed φ viewed on $E \times \mathbb{R}$ is in $C_u(E \times \mathbb{R})$, and moreover that if φ is a smooth vector in $C_\infty(\mathbb{R})$ for translation then it is a smooth vector also in $C_u(E \times \mathbb{R})$ for ρ^α , and thus $\Theta^\alpha(\varphi)$ will be in B^∞ . We need to use this fact in order to check the appropriate Weyl relations for the u_α 's. (A simple calculation at the level of functions, given below, shows what these relations should be. But we must continue to work in the framework of [5] in order to see that all is compatible with respect to the operator norms which are present.)

Since we will now vary E , we need to make E explicit in our notation. So we will now reserve A for $C_\infty(G \times V)$, and write ${}^E A$ for $C_\infty(E \times V)$, with similar notation for B . We have the evident restriction map, R^E , from $C_\infty(G \times V)$ to $C_b(E \times V)$, which is a morphism from A to ${}^E A$. Since E is open, the range of R^E will contain ${}^E A$, but it will usually contain much more. Clearly R^E is equivariant for τ acting on both A and ${}^E A$. Thus R^E determines a morphism, R_J^E , from A_J to ${}^E A_J$, by theorem 3.1 of [7].

Since E is open, each function in $C_\infty(E \times V)$ can be viewed as a function in $C_\infty(G \times V)$ by setting it equal to 0 off $E \times V$. In this way ${}^E A$ is an ideal in A , carried into itself by τ . Thus ${}^E A_J$ is an ideal in A_J by proposition 5.9 of [5]. In particular, for $f \in {}^E A^\infty$, viewed as in A^∞ , we can consider $R_J^E(u_\alpha \times_J f) = R_J^E(u_\alpha) \times_J R_J^E(f)$. Now if E is bounded, $R^E(u_\alpha)$ is a smooth vector in ${}^E B$. This is exactly our reason for restricting to bounded E . For then we can calculate the deformed product directly by our oscillatory integrals. To lighten our notation we will omit the R^E but work on E . Then we have:

3.1 Lemma. *For bounded E , for $f \in {}^E A^\infty$, and for $\alpha \in V'$, we have*

$$(u_\alpha \times_J f)(T, q) = e(q \cdot \alpha) f(T, q + T J T^t \alpha)$$

for $(T, q) \in G \times V$, in the sense that $u_\alpha \times_J f$ as element of ${}^E A_J$ is represented by the function on the right.

Proof. For $(T, q) \in E \times V$ we have

$$\begin{aligned} (u_\alpha \times_J f)(T, q) &= \int_{V'} \int_V e(\alpha(q - T J T^t p)) f(T, q - v) e(v \cdot p) dv dp \\ &= e(q \cdot \alpha) \int_{V'} \int_V e((v + T J T^t \alpha) \cdot p) f(T, q - v) dv dp \\ &= e(q \cdot \alpha) f(T, q + T J T^t \alpha), \end{aligned}$$

where we have used the Fourier inversion formula in the last step. Thus $R_J^E(u_\alpha \times_J f)$ is given by the function on the right. But R_J^E is injective by proposition 5.8 of [5]. Q.E.D.

3.2 Theorem. *For $\alpha, \beta \in V'$ we have*

$$u_\alpha \times_J u_\beta = e(Q_{\alpha\beta})u_{\alpha+\beta}$$

in $M(A_J)$, where $Q_{\alpha\beta}$ is defined on $G \times V$ by

$$Q_{\alpha\beta}(T, q) = \beta(TJT^t\alpha).$$

Proof. We remark first that $e(Q_{\alpha\beta})$ is invariant under τ since $Q_{\alpha\beta}$ is independent of q . Thus $e(Q_{\alpha\beta})$ is a smooth vector in B , whose deformed product with any function in B^∞ is just the pointwise product (by corollary 2.13 of [5]). (In particular, all the $e(Q_{\alpha\beta})$'s commute among themselves for the deformed product, as do thus the $Q_{\alpha\beta}$'s.) In this way $e(Q_{\alpha\beta})u_{\alpha+\beta}$ has meaning. Now for any bounded E and for any $f \in {}^E A^\infty$ we have (omitting R^E from our notation)

$$\begin{aligned} (u_\alpha \times_J (u_\beta \times_J f))(T, q) &= e(q \cdot \alpha)(u_\beta \times_J f)(T, q + TJT^t\alpha) \\ &= e(q \cdot \alpha)e((q + TJT^t\alpha) \cdot \beta)f(T, q + TJT^t\alpha + TJT^t\beta) \\ &= e(\beta(TJT^t\alpha))(u_{\alpha+\beta} \times_J f)(T, q), \end{aligned}$$

for $(T, q) \in E \times V$. But ${}^E A_J$ is an ideal in A_J , and so, using associativity, we have

$$(u_\alpha \times_J u_\beta) \times_J f = (e(Q_{\alpha\beta})u_{\alpha+\beta}) \times_J f.$$

But the union of the ${}^E A$'s is a dense τ -invariant ideal in A . It is easily seen from the proof of proposition 2.17 of [5] that the smooth elements of this dense ideal are dense in A_J . Consequently $u_\alpha \times_J u_\beta = e(Q_{\alpha\beta})u_{\alpha+\beta}$ as desired. \square

This theorem shows that the unbounded operators \mathbf{q}_α affiliated with A_J have an integrated Weyl form, so that they give a “regular” realization in the sense of [4], satisfying

$$[\mathbf{q}_\alpha, \mathbf{q}_\beta] = Q_{\alpha\beta}.$$

The operators $Q_{\alpha\beta}$ are affiliated in the evident way with $C_\infty(G)$, and so with A_J through the evident morphism of $C_\infty(G)$ to $C_\infty(G \times V)$. For the case considered in [4] in which V is Minkowski space, G is the Lorentz group, and J is the standard symplectic matrix as in 3.28 of [4], the above operators will satisfy the conditions desired in [4].

4. THE SPACE-TIME ALGEBRA

In the situation discussed in the previous section we notice that the dependence of $Q_{\alpha\beta}$ on T is really only on TJT^t , and that in Lemma 2 the T -dependence of the product with u_α is only on TJT^t . But the TJT^t 's comprise the orbit, Σ , of J under G . This indicates that our algebra $C_\infty(G \times V)$ is unnecessarily large, and that we should be working on $\Sigma \times V$. We did not do this initially because there is no convenient action of V there which we could use so

as to apply the constructions of [5]. But there is a simple device by which we can now pass to $\Sigma \times V$.

Let H denote the stability subgroup of J in G , consisting of those $S \in G$ such that $SJS^t = J$ (so that Σ is identified with G/H). Since the set of S 's in $\text{Aut}(V)$ which satisfy this relation is just the symplectic group for J , we see that H is just the intersection of G with this symplectic group. Define a (free and proper) right action, γ , of H on $G \times V$ by

$$\gamma_S(T, q) = (TS, q),$$

and let γ also denote the corresponding action on $B = C_u(G \times V)$. This action does not commute with τ in general. But for $f \in B$ we have

$$\begin{aligned} (\tau_x(\gamma_S f))(T, q) &= f(TS^{-1}, q - Tx) \\ &= (\gamma_S(\tau_{Sx} f))(T, q), \end{aligned}$$

that is, $\tau_x \gamma_S = \gamma_S \tau_{Sx}$. But this is exactly the condition (together with $(SJS^t = J)$ which according to proposition 10.4 of [5] insures that γ_S determines an automorphism of B_J . Thus γ gives a right action (which we still denote by γ) of H on B_J . We do not need to explore here the strong continuity of this action, since what we are interested in is the fixed-point subalgebra, D_J . Thus for this purpose we can treat H as a discrete group.

At the function level it is clear that there are many elements in D_J . Any $f \in C_c(\Sigma \times V)$ which is smooth in the V -direction will lift to an element of D_J . It is clear from Lemma 3.1 that taking products with any u_α will carry D_J into itself, so that the u_α 's can be viewed as multipliers of D_J . They will still satisfy the Weyl relations

$$u_\alpha \times_J u_\beta = e(Q_{\alpha\beta}) u_{\alpha+\beta},$$

but now we can view $Q_{\alpha\beta}$ as the function on Σ (or $\Sigma \times V$) defined by

$$Q_{\alpha\beta}(\sigma) = \beta(\sigma(\alpha)).$$

This means that we still have the unbounded operators \mathbf{q}_α , but now affiliated with D_J . They still satisfy

$$[\mathbf{q}_\alpha, \mathbf{q}_\beta] = Q_{\alpha\beta}$$

as above. It is easily seen that we obtain:

4.1 Theorem. *The C^* -algebra D_J together with its affiliated unbounded operators \mathbf{q}_α and corresponding unitary operators u_α , is the universal C^* -algebra for “regular realizations” of the commutation relations over Σ , that is, integrable representations for which*

$$[\mathbf{q}_\alpha, \mathbf{q}_\beta] = Q_{\alpha\beta},$$

where the $Q_{\alpha\beta}$'s commute among themselves and with the \mathbf{q}_α 's, and for which the joint spectrum of the $Q_{\alpha\beta}$'s is a subset of Σ .

When V is Minkowski space, etc, then D_J is essentially the C^* -algebra of [4] for the space-time uncertainty relations.

We could continue by applying other parts of [5], for example chapter 8 to discuss the continuous field aspect which is discussed in [4], or chapter 9 to discuss the semi-classical limit. But this is all relatively straightforward, and so we will stop our discussion here.

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